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ON THE WEAK LAW OF LARGE NUMBERS FOR D(0,1).(U)
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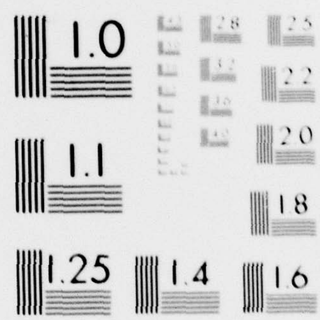

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6 ON THE WEAK LAW OF LARGE NUMBERS FOR D[0,1].

by

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University of South Carolina
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Abstract

Weak laws of large numbers are obtained for random elements in D[0,1] where the convergence is in the sup-norm topology. For identically distributed random elements satisfying a compact integral condition, the weak law of large numbers holding pointwise is shown to be necessary and sufficient for the weak law of large numbers. In addition to a discussion of the compact integral condition, a weak law of large numbers is obtained for monotone increasing random elements, and convergence of weighted sums of independent, identically distributed random elements is obtained.

AMS Classification Number: Primary 60B05, 60F15; Secondary 60G99.

Key words and phrases: Skorohod topology, laws of large numbers, convergence in probability, partitions, and identical distributions.

1. Introduction and Preliminaries. Let $D = D[0,1]$ denote the space of real-valued functions on $[0,1]$ which are right-continuous and possess left-hand limits at each $t \in [0,1]$. Let the linear space D be equipped with the topology generated by the Skorohod metric d and let $\|\ \|$ denote the uniform norm, $\|x\| = \sup_{0 \le t \le 1} |x(t)|$ for $x \in D$ (see Billingsley (1968), pages 109-153 for detailed topological and probabilistic properties of D). With the Skorohod metric topology, the linear space D is separable but not a linear metric space (addition is not continuous). However, the major obstacle in developing laws of large numbers is the absence of local convexity for D with the Skorohod topology.

Ranga Rao (1963) proved a strong law of large numbers for independent, identically distributed random elements in D while Sethuraman (1965) proved a large deviation result for independent, identically distributed random elements in D. Their methods of proof used truncation to a compact set and the following lemma.

Lemma 1 (Ranga Rao (1963)): Let X be a random element in D with $E\|X\| < \infty$. For each $\varepsilon > 0$ there exists a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ of $[0,1]$ such that

$$\max_{1 \le i \le m} \quad \sup_{t_{i-1} \le s, t < t_i} \quad E|X(t) - X(s)| \le \varepsilon.$$

To compensate for the absence of local convexity, Taylor and Daffer (1978) & (1979) required the random elements to be convex tight and obtained laws of large numbers in the Skorohod topology. In this paper a weak law of large numbers is obtained in the uniform norm topology for identically distributed random elements satisfying the following compact integral condition.

<u>Property MT</u>: A random element X in D[0,1] has property MT if for each $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists a compact set K and a partition $0 = t_0 < \ldots < t_m = 1$ such that

$$E[\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]}] < \varepsilon_1$$

and where $P[X \in K] > 1 - \varepsilon_2$.

Since convex tightness and a finite first moment is sufficient for property MT, the convergence in the $\|\ \|$ - topology significantly improves the laws of large numbers for convex tight random elements. A brief discussion of property MT will be included, and a weak law for the convergence of weighted sums of random elements will also be obtained in Section 3.

<u>2. A Weak Law of Large Numbers</u>. Given a partition $0 = t_0 < t_1 < \ldots < t_m = 1$, for notational convenience let

$$Tx = \sum_{i=0}^{m} x(t_i) I_{[t_i, t_{i+1})}(t) \tag{2.1}$$

where $x = x(t) \in D[0,1]$ and $[t_m, t_{m+1}) = \{1\}$. It is easy to see that T is a linear, Borel measurable (with respect to the Skorohod topology) function from D into D and that $\|Tx\| \le \|x\|$ for all $x \in D$.

<u>Theorem 1</u>: Let $\{X_n\}$ be identically distributed random elements in D such that $E\|X_1\| < \infty$ and let X_1 have property MT. Then

$$\frac{1}{n} \sum_{k=1}^{n} X_k(t) \to EX_1(t) \text{ in probability}$$

for each $t \in [0,1]$ if and only if

$$\left\|\frac{1}{n} \sum_{k=1}^{n} X_k - EX_1\right\| \to 0 \text{ in probability.}$$

<u>Proof</u>: The "if" part is immediate. For the "only if" part let $\varepsilon > 0$ and $0 < \eta \le 1$ be given. Since $E||X_1|| < \infty$ and X_1 has property MT, there exists a compact set K such that

$$E(||X_1||I_{[X_1 \notin K]}) < \frac{\varepsilon\eta}{12} \tag{2.2}$$

and a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ such that

$$E[\max_{0\le i\le m-1} \sup_{t_i\le t<t_{i+1}} |X_1(t)-X_1(t_i)|I_{[X_1 \in K]}] < \frac{\varepsilon\eta}{12}. \tag{2.3}$$

For each n

$$\begin{aligned} &P[||\tfrac{1}{n}\textstyle\sum_{k=1}^n X_k - EX_1|| > \varepsilon] \\ &\quad \le P[||\tfrac{1}{n}\textstyle\sum_{k=1}^n TX_k - E(TX_1)|| > \tfrac{\varepsilon}{4}] \\ &\qquad + P[||\tfrac{1}{n}\textstyle\sum_{k=1}^n X_k - \tfrac{1}{n}\sum_{k=1}^n TX_k|| > \tfrac{\varepsilon}{2}] \\ &\qquad + P[||EX_1 - E(TX_1)|| > \tfrac{\varepsilon}{4}]. \end{aligned} \tag{2.4}$$

The third term of (2.4) is zero since

$$\begin{aligned} &||EX_1 - E(TX_1)|| \\ &\quad \le E(||X_1 - TX_1||I_{[X_1 \in K]}) + E(||X_1 - TX_1||I_{[X_1 \notin K]}) \\ &\quad < E[\max_i \sup_{t_i\le t<t_{i+1}} |X_1(t)-X_1(t_i)|I_{[X_1 \in K]}] + \frac{2\varepsilon\eta}{12} \\ &\quad < \frac{\varepsilon}{12} + \frac{2\varepsilon}{12} = \frac{\varepsilon}{4} \end{aligned} \tag{2.5}$$

by (2.2) and (2.3). Using Markov's inequality and the identical distributions for the second term of (2.4),

$$P[||\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}TX_k||>\frac{\varepsilon}{2}]$$

$$\le \frac{2}{\varepsilon n}\sum_{k=1}^{n}E||X_k-TX_k||$$

$$\le \frac{2}{\varepsilon}(E[\max_i \sup_{t_i\le t<t_{i+1}} |X_1(t)-X_1(t_i)|I_{[X_1\in K]}]$$

$$+ 2E(||X_1||I_{[X_1\notin K]}))$$

$$< \frac{2}{\varepsilon}(\frac{\varepsilon\eta}{12}+\frac{2\varepsilon\eta}{12}) = \frac{\eta}{2}. \tag{2.6}$$

The weak law of large numbers holds for $\{X_k(t_i):\ k\ge 1\}$ with $i = 0,1,\ldots,m$ by hypothesis. Thus, for the first term of (2.4),

$$P[||\frac{1}{n}\sum_{k=1}^{n}TX_k-E(TX_1)||>\frac{\varepsilon}{4}]$$

$$\le P[\max_{0\le i\le m}|\frac{1}{n}\sum_{k=1}^{n}X_k(t_i)-E(X_1(t_i))|>\frac{\varepsilon}{4}]$$

$$< \frac{\eta}{2} \tag{2.7}$$

for all $n \ge N(\varepsilon,\eta)$. Combining (2.4), (2.5), (2.6), and (2.7) yields

$$P[||\frac{1}{n}\sum_{k=1}^{n}X_k-EX_1||>\varepsilon]<\frac{\eta}{2}+\frac{\eta}{2}=\eta$$

for all $n\ge N(\varepsilon,\eta)$. ///

A random element X in D is said to be <u>convex tight</u> if for each $\varepsilon > 0$ there exists a compact, convex set K such that $P[X\in K] > 1-\varepsilon$. Characterizations of convex tightness were given by Daffer and Taylor (1979).

The following lemma shows that convex tightness will imply the property MT.

<u>Lemma 2</u>: If a random element X in D is convex tight and $E||X|| < \infty$, then X has property MT.

Proof: Given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, choose K convex and compact such that $P[X \in K] > 1 - \varepsilon_2$. By Theorem 6 of Daffer and Taylor (1979),

$$\{t \in [0,1]: \sup_{x \in K} |x(t)-x(t-0)| > \frac{\varepsilon_1}{2}\}$$

is finite $(=\{s_1,\ldots,s_r\})$. Let

$$\{t_0,t_1,\ldots,t_m\} = \{s_1,\ldots,s_r\} \cup \{0,\frac{1}{k},\ldots,\frac{k}{k}\}.$$

Then, $k \le m \le k+r$, and

$$\limsup_{m\to\infty} \{\max_i \sup_{t_i \le t < t_{i+1}} |x(t)-x(t_i)|\} \le \frac{\varepsilon_1}{2}.$$

By the bounded convergence theorem

$$\limsup_{m\to\infty} E[\max_i \sup_{t_i \le t < t_{i+1}} |X(t)-X(t_i)| I_{[X \in K]}] \le \frac{\varepsilon_1}{2}$$

$$\le E[\limsup_{m\to\infty} \max_i \sup_{t_i \le t < t_{i+1}} |X(t)-X(t_i)| I_{[X \in K]}] \le \frac{\varepsilon_1}{2}.$$

Thus, there exists m such that

$$E[\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} |X(t)-X(t_i)| I_{[X \in K]}] < \varepsilon_1. \qquad ///$$

From Lemma 2 it follows that the weak law of large numbers with convergence in the $\|\ \|$-topology holds for many situations. The most standard condition may be pointwise uncorrelated identically distributed random elements which satisfy a convexity condition and hence yields an appropriate partition of [0,1].

3. Related Results. An example of a random element which does not have property MT is somewhat (by Lemma 2) related to nonseparable support with respect to the $\|\ \|$-topology. Let $\Omega = [0,1]$, $\mathcal{A}$ be the Borel subsets, and $X(t) = I_{[\omega,1]}(t)$. The uniform probability measure on [0,1] is assumed, that is, $P[X \in \{I_{[s,1]}: 0 \le s_1 \le s \le s_2 \le 1\}] = s_2 - s_1$. Let K be any compact set such that $P[X \notin K] < \varepsilon$. For any partition $\{t_0, \ldots, t_m\}$ of [0,1]

$$\sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| = 1 \quad \text{for } \omega \in (t_i, t_{i+1}).$$

Thus,

$$\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]} = 1$$

if and only if $\omega \in [X \in K] \cap (\bigcup_{i=0}^{m-1} (t_i, t_{i+1}))$. Hence,

$$\begin{aligned} E[&\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]}] \\ &= P[[X \in K] \cap (\bigcup_{i=0}^{m-1} (t_i, t_{i+1}))] \\ &= P[X \in K] > 1 - \varepsilon. \end{aligned} \tag{3.1}$$

By (3.1) X does not have property MT.

The property MT is certainly not necessary for the weak law of large numbers in D[0,1]. Note that X is monotone increasing with probability one. The following weak law of large numbers for monotone increasing random elements can be proved in the same manner as Theorem 2 of Daffer and Taylor (1979).

Theorem 2: Let $D^{\dagger}$ denote the cone of nondecreasing elements of D. Let $\{X_n\}$ be a sequence of random elements in D such that $X_n \in D^{\dagger}$ with probability one and $EX_n = EX_1$ for each n. Then,

$$\left|\frac{1}{n}\sum_{k=1}^{n} X_k(t) - EX_1(t)\right| \to 0 \text{ in probability}$$

for each $t \in [0,1]$ if and only if

$$\left|\left|\frac{1}{n}\sum_{k=1}^{n} X_k - EX_1\right|\right| \to 0 \text{ in probability.}$$

The major use of property MT in obtaining the weak law of large numbers for D is in developing Inequality (2.6). It is instructive (for applications) to consider

$$P\left[\left|\left|\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} TX_k\right|\right| > \frac{\varepsilon}{2}\right]$$

$$= P\left[\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} \left|\frac{1}{n}\sum_{k=1}^{n} (X_k(t) - X_k(t_i))\right| > \frac{\varepsilon}{2}\right]$$

when the $\{X_k(t)\}$ are stochastic processes with independent increments. For these applications, Levy's inequalities may yield Inequality (2.6).

The generality of the pointwise condition is balanced by a compact, smooth type condition (property (MT)). Additional probability structure on the random elements can eliminate the need for property MT as is illustrated in Theorem 3 for the convergence of weighted sums. Only the essential steps of the proof for Theorem 3 will be listed for comparison with the proof of Theorem 1.

Theorem 3: Let $\{X_n\}$ be independent, identically distributed random elements in D such that $E||X_1|| < \infty$. Let $\{a_{nk}\}$ be a double array of constants such

that (i) $\sum_{k=1}^{\infty}|a_{nk}| \le 1$ for each n and (ii) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1$. A necessary and sufficient condition that

$$||\sum_{k=1}^{n} a_{nk}X_k - EX_1|| \to 0 \text{ in probability}$$

is that $\max_{1\le k\le n} |a_{nk}| \to 0$.

Outline of Proof: Since real numbers can be identified with constant functions in D, $\max_{1\le k\le n} |a_{nk}| \to 0$ is necessary. For sufficiency pick K compact such that (2.2) holds. Choose δ so that $\alpha \le t < \beta < \alpha+\delta$ implies that $|x(t) - x(\alpha)| \le |x(\beta-0)-x(\alpha)| + \frac{\varepsilon\eta}{8}$ uniformly for $x \in K$. In place of (2.3), Lemma 1 provides a partition such that

$$\max_{0\le i\le m-1} \sup_{t_i\le t, s<t_{i+1}} E|X(s)-X(t)| \le \frac{\varepsilon\eta}{12},$$

and it can be assumed that $t_{i+1} - t_i < \delta$ for all $0\le i\le m-1$. For (2.5)

$$||EX_1 - ETX_1||$$

$$= \max_{0\le i\le m-1} \sup_{t_i\le t<t_{i+1}} E|X(t)-X(t_i)| \le \frac{\varepsilon\eta}{12} < \frac{\eta}{6}. \quad (3.2)$$

Using the independence of $\{X_k(t_{i+1}-0) - X_k(t_i): k = 1,2,3,\ldots\}$ for each i, (2.6) becomes

$$P[||\sum_{k=1}^{n} a_{nk}X_k - \sum_{k=1}^{n} a_{nk}TX_k|| > \frac{\varepsilon}{2}]$$

$$\le P[\max_{0\le i\le m} \sup_{t_i\le t<t_{i+1}} |\sum_{k=1}^{n} a_{nk}(X_k(t)-X_k(t_i))I_{[X_k \in K]}| > \frac{\varepsilon}{4}]$$

$$+ P[\sum_{k=1}^{n}|a_{nk}|\ ||X_k-TX_k||I_{[X_k \notin K]} > \frac{\varepsilon}{4}]$$

$$\le P[\max_{0\le i\le m-1}\sum_{k=1}^{n}|a_{nk}||X_k(t_{i+1}-0)-X_k(t_i)|I_{[X_k\in K]}>\frac{\varepsilon}{8}]$$

$$+\frac{4}{\varepsilon}\sum_{k=1}^{n}|a_{nk}|2E(||X_k||I_{[X_k\notin K]})$$

$$\le \sum_{i=0}^{m-1}P[\sum_{k=1}^{n}|a_{nk}|(|X_k(t_{i+1}-0)-X_k(t_i)|I_{[X_k\in K]}$$

$$-E(|X_k(t_{i+1}-0)-X_k(t_i)|I_{[X_k\in K]})>\frac{\varepsilon}{8}-\frac{\varepsilon}{12}]$$

$$+\frac{8}{\varepsilon}\cdot\frac{\varepsilon\eta}{12}. \tag{3.3}$$

Since the random variables $\{|X_k(t_{i+1}-0)-X_k(t_i)|I_{[X_k\in K]}: k = 1,2,\ldots\}$ are independent, identically distributed random elements, the first term can be made less than $\frac{\eta}{6}$. ///

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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